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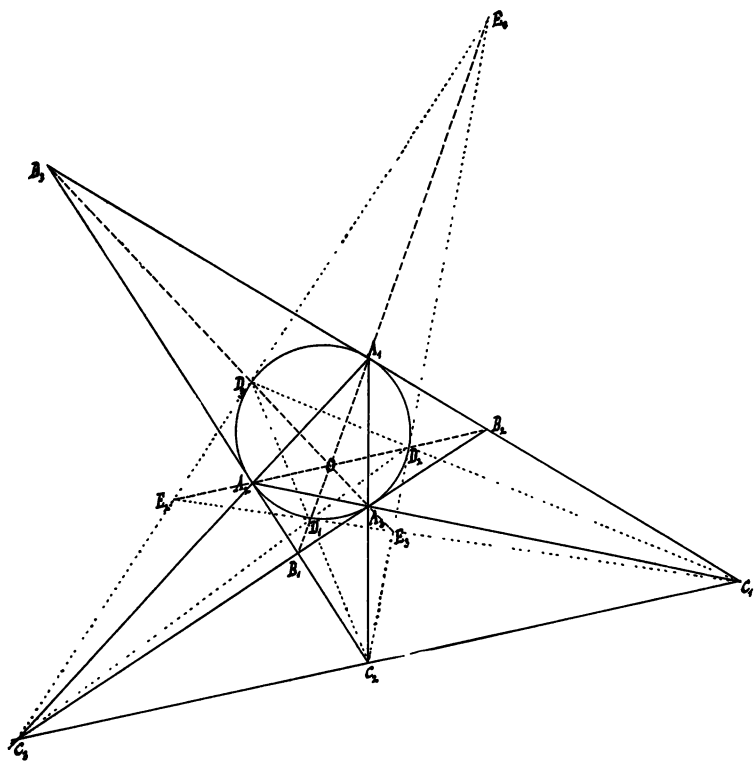
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A USEFUL DIAGRAM FOR EXAMPLES IN MODERN ANALYTIC GEOMETRY.

By PROFESSOR H. MASCHKE.

In the following brief article I discuss, from different points of view, a certain diagram, which leads to a great number of examples and exercises in modern analytic geometry.

Let A_1 be a point on the conic

$$x_1x_2 + x_2x_3 + x_3x_1 = 0.$$

Designate its trilinear coördinates x_1, x_2, x_3 by a, β, γ , respectively, and write briefly

$$A_1 = (a, \beta, \gamma).$$

Then the points

$$A_2 = (\beta, \gamma, a) \text{ and } A_3 = (\gamma, a, \beta)$$

will lie on the same conic. They form with A_1 a triangle which shall be denoted by $[a, \beta, \gamma]$.

According to Pascal's theorem the intersections C_1, C_2, C_3 of the tangents to the conic at A_1, A_2, A_3 and the respective opposite sides of the triangle lie on a straight line ω .

The equation of the tangent at A_1 is

$$(\beta + \gamma)x_1 + (\gamma + a)x_2 + (a + \beta)x_3 = 0,$$

and the coördinates of $C_1=(\beta-\gamma, \gamma-\alpha, \alpha-\beta)$. The coördinates of C_2 and C_3 are obtained from those of C_1 by a cyclic permutation of α, β, γ . Hence the line ω has the equation

$$x_1 + x_2 + x_3 = 0,$$

and its pole O the coördinates $(1, 1, 1)$.

The line ω is, therefore, independent of the selection of the point A_1 on the conic. All the triangles $[a, \beta, \gamma]$ have one and the same Pascal-line ω , and every triangle $[a, \beta, \gamma]$ determines on ω a triple of points C_1, C_2, C_3 .

If we select both the conic section and the line ω arbitrarily (not, however, intersecting in real points), then every triangle satisfying the conditions of the diagram is a triangle $[a, \beta, \gamma]$. In other words: If we take as triangle of reference one of the inscribed triangles and the line ω as unit-line (or its pole O as unit-point) and if further we choose the point $A_1=(a, \beta, \gamma)$ arbitrarily on the conic, then the coördinates of A_2 and A_3 are found to be (β, γ, a) and (γ, a, β) .

Every point-triple C_1, C_2, C_3 is determined by one of its points, C_1 for instance, just as every triangle $[a, \beta, \gamma]$ is determined by one of its vertices, *e. g.* A_1 . Every point C_1 , however, while it determines C_2 and C_3 uniquely, determines two triangles $[a, \beta, \gamma]$, namely, A_1, A_2, A_3 and D_1, D_2, D_3 , corresponding to the two tangents which can be drawn from C_1 to the conic.

The coördinates of all the lines and points shown in the diagram are all given by simple expressions in terms of α, β, γ and can easily be deduced by always using the relation

$$\alpha\beta + \beta\gamma + \gamma\alpha = 0$$

for simplification.

As examples I give the coördinates of the points

$$\begin{aligned} B_1 &= (\alpha - \beta - \gamma, \beta - \gamma - \alpha, \gamma - \alpha - \beta), \\ D_1 &= (2\beta + 2\gamma - \alpha, 2\gamma + 2\alpha - \beta, 2\alpha + 2\beta - \gamma), \\ E_1 &= (\beta + \gamma - 5\alpha, \gamma + \alpha - 5\beta, \alpha + \beta - 5\gamma). \end{aligned}$$

The line ω meets the conic in the two imaginary points $1, \varepsilon, \varepsilon^2$ and $1, \varepsilon^2, \varepsilon$, where

$$\varepsilon = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}.$$

These two imaginary points ω_1 and ω_2 , together with any point-triple C_1, C_2, C_3 form a so-called cyclic-projective system of points. That means the following three ranges of five points are projective:

$$(\omega_1, C_1, C_2, C_3, \omega_2), \quad (\omega_1, C_2, C_3, C_1, \omega_2), \quad (\omega_1, C_3, C_1, C_2, \omega_2).$$

Since the two triangles B_1, B_2, B_3 and E_1, E_2, E_3 are circumscribed about one and the same conic, the six vertices B and E must lie on a conic. The equation of this conic is

$$x_1^2 + x_2^2 + x_3^2 + 3(x_1x_2 + x_2x_3 + x_3x_1) = 0.$$

To this conic the triangle B_1, B_2, B_3 (or E_1, E_2, E_3) bears the same relation as the original triangle A to the original conic, *i. e.* the tangents to the conic at B_1, B_2, B_3 (or E_1, E_2, E_3) pass through C_1, C_2, C_3 , respectively.

Likewise the conic

$$x_1^2 + x_2^2 + x_3^2 - 2(x_1x_2 + x_2x_3 + x_3x_1) = 0$$

is inscribed in the two triangles A and D . The lines joining the contact points of these two triangles pass again through C_1, C_2, C_3 . In this way we obtain an infinite chain of conics Γ . All these conics Γ pass through the same points ω_1, ω_2 on the line ω , and since the point O is the pole of ω with respect to every conic, the conics Γ are tangent to each other at ω_1 and ω_2 .

If we transform by projection the original conic into a circle and simultaneously the line ω into the line at infinity, which is always possible, then the conics Γ are projected into conics passing through the two circular points at infinity where they are tangent to each other, *i. e.* into concentric circles. The inscribed and circumscribed triangles of the diagram are transformed into equilateral triangles.

From this system of concentric circles with inscribed and circumscribed triangle all the properties of the diagram involving the conics Γ can now easily be deduced.

If we take as original conic an ellipse and as line ω the line at infinity, the point O becomes the center of the ellipse and also the point of intersection of the medians of every triangle A_1, A_2, A_3 , and it can easily be shown not only that all these triangles are of equal area, but also that this area represents the maximum value of the areas of all triangles inscribed into the ellipse.

THE UNIVERSITY OF CHICAGO, November, 1905.

A PERFECT MAGIC SQUARE.

By PROFESSOR F. ANDEREGG.

In the accompanying magic square, the sum of the numbers in any line, column, or diagonal is the same (2056). Instead of continuous lines, any four numbers can be taken from the first and third groups of four numbers, and any four from the second and fourth groups in the first line, and the corresponding numbers in the last line. Similar combinations can be made for the second and fifteenth lines, the third and fourteenth, etc. Similar combinations can also be made for the columns.

Instead of a complete diagonal two incomplete diagonals can be taken on